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ON WRAPPING OF NEW WEIBULL PARETO DISTRIBUTION

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ABSTRACT
In many scientific experiments the data have been observed exhibiting periodic or cyclic behaviour. To deal with such data and to perform statistical analysis many Circular models were developed from the existing linear distributions by adopting variety of techniques like wrapping, inverse stereographic projections, rising sun function etc. Also a good number of circular models for linear life testing models were constructed using the technique of wrapping. In this article an attempt is made to construct a circular model for the three-parameter Weibull–Pareto distribution namely 'New Weibull–Pareto Distribution' using the method of wrapping. The Probability density function, distribution function and characteristic function are derived for this Wrapped New Weibull Pareto Distribution. The Trigonometric moments and through them some of the important population characteristics for this wrapped New Weibull Pareto Distribution are computed and tabled.

Keywords: Circular Distribution, Wrapping, Trigonometric Moments, New Weibull Pareto.

1 INTRODUCTION

Dattatreya Rao et al (2007) constructed Wrapped Lognormal, Wrapped Logistic, Wrapped Weibull, and Wrapped Extreme Value Distributions. Ramabhadra Sarma et al (2009) derived characteristic function of Wrapped Half Logistic and Wrapped Binormal Distribution. Mardia and Jupp (2000) gave expressions for population characteristics such as variance, standard deviation, skewness, kurtosis etc. for circular distributions. Girija et al (2010) introduced new construction procedures of constructing Circular models calling Rising Sun Circular models and studied M L estimation parameters of Cardioid distribution from complete samples.

Contributing to this work, an attempt is made here to derive a new circular model for the New Weibull Pareto distribution using the method of wrapping. Wrapping is a technique which reduces a linear variable to it's modulo 2π. The density, distribution function, characteristic function for the wrapped New Weibull Pareto distribution are derived and using the trigonometric moments, important population characteristics for the proposed Wrapped New Weibull Pareto distribution are also computed.

This paper is organised as follows. Section 2 describes the Circular probability distribution and the methodology of wrapping a linear probability distribution. Section 3 defines the proposed wrapped New Weibull Pareto distribution, and presents the graphs of probability density, distribution and characteristic functions for different values of parameters. Important population characteristics for the wrapped New Weibull Pareto distribution are also tabulated. Section 4 summarises the findings of this study.

For this paper software MATLAB is used for all the computations and for plotting of graphs.

2 CIRCULAR PROBABILITY DISTRIBUTION

A circular random variable in a continuous circular distribution $g:[0,\ 2\pi)\to \mathbb{R}$ is said to be following a circular probability density function of g (θ) if and only if g has the following basic properties

(i) $g(\theta) \geq 0, \quad \forall \theta$ … (1)

(ii) $\int_0^{2\pi} g(\theta)\, d\theta = 1$ … (2)

(iii) $g(\theta) = g(\theta + 2k\pi)$ g is periodic , for any integer k (Mardia,2000) … (3)

a) Method of Wrapping

If X is a random variable defined on $\mathbb{R}$, then the corresponding circular random variable X_w is defined by the modulo 2π reduction.

$$X_W \equiv X(\text{mod } 2\pi)$$

If f (x) is the probability density function (pdf) of the linear random variable X then for the circular random variable $X_{w,}$ the corresponding pdf, g(θ) is defined as,

$$g(\theta) = \sum_{k=-\infty}^{\infty} f(\theta + 2k\pi)\text{, where } \theta \in [0, 2\pi)$$

It can be verified that $g(\theta)$ with total probability concentrated on the unit circle $\{(\cos\theta, \sin\theta) / 0 \leq \theta < 2\pi\}$ in the plane and satisfies the properties (1) to (3) above.

Also the characteristic function for X_w given its distribution function $F(\theta)$ is given by

$$\varphi_\theta(t) = E(e^{it\theta}) = \int_0^{2\pi} e^{it\theta} dF(\theta) = \rho_t e^{i\mu_t} \quad t \in$$

It is clear from the above that whenever $\varphi(t) \neq 0$, $e^{2\pi it} = 1$ (Mardia 2000). Implies $\varphi(t)$ can only be defined for integer values of t. Also the characteristic function for the wrapped distribution is $\varphi(p) = \varphi_p$ and is defined as

$$\varphi_\theta(p) = E(e^{ip\theta}) = \int_0^{2\pi} e^{ip\theta} dF(\theta) = \rho_p e^{i\mu_p}\text{ , p} \in \mathbf{Z} \quad \text{Also } \varphi_0 = 1, \quad \bar{\varphi}_p = \varphi_{-p},$$

b) Trigonometric moments

φ_p , the pth trigonometric moment is value of the characteristic function φ_t at $t = p$.

The real part and the imaginary part of φ_p are trigonometric moments α_p and β_p respectively and are denoted as

$$\alpha_p = E(\cos p\theta), \quad \beta_p = E(\sin p\theta) \quad \text{Where} \quad p \in \mathbf{Z}$$

3 WRAPPED NEW WEIBULL PARETO DISTRIBUTION (WNWP)

A linear random variable X is said to follow a three parameter New Weibull Pareto Distribution if the distribution function of X is given by

$$F(X) = 1 - e^{-\lambda\left(\frac{x}{\delta}\right)^{C}}$$

Where ***c*** is the shape parameter and *λ* and *δ* are the scale parameters and

$0 < x < \infty$ and $c > 0$, $\lambda > 0$ and $\delta > 0$

Hence the pdf of New Weibull Pareto Distribution is

$$f(x) = \frac{c.\lambda}{\delta}\left(\frac{x}{\delta}\right)^{c-1} e^{-\lambda\left(\frac{x}{\delta}\right)^{c}}$$

Where $0 < x < \infty$ and $c > 0$, $\lambda > 0$ *and* $\delta > 0$

Here if δ = 1, this NWP distribution reduces to Weibull distribution and if δ = 1 and *c* = 1 this distribution reduces to Exponential distribution.

a) Probability density function for WNWP distribution

Applying the method wrapping the pdf for WNWP distribution $g(\theta)$ can be written as

$$g(\theta) = \sum_{k=0}^{\infty} \frac{c.\lambda}{\delta}\left(\frac{\theta + 2k\pi}{\delta}\right)^{c-1} e^{-\lambda\left(\frac{\theta+2k\pi}{\delta}\right)^{c}}$$

Where $\theta \in [0, 2\pi)$ and $c > 0$, $\lambda > 0$ and $\delta > 0$

The graph depicting the linear representation of the pdf of WNWP Distribution for different values of ***c, δ***, keeping the value for the parameter **λ** at 2.0 is as follows:

Figure 1: PDF of WNWP distribution (Linear Representation)

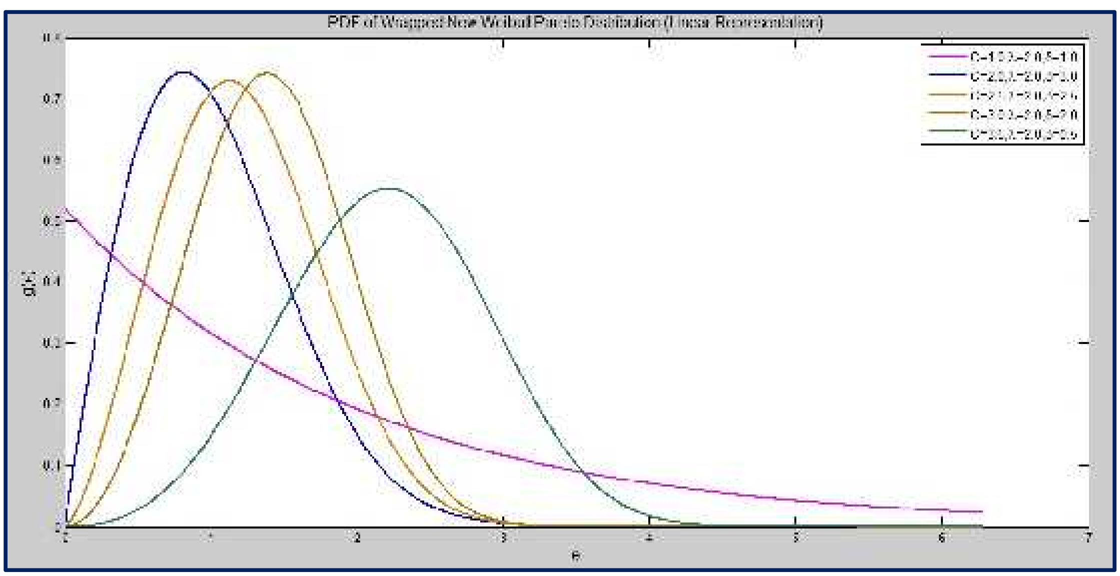
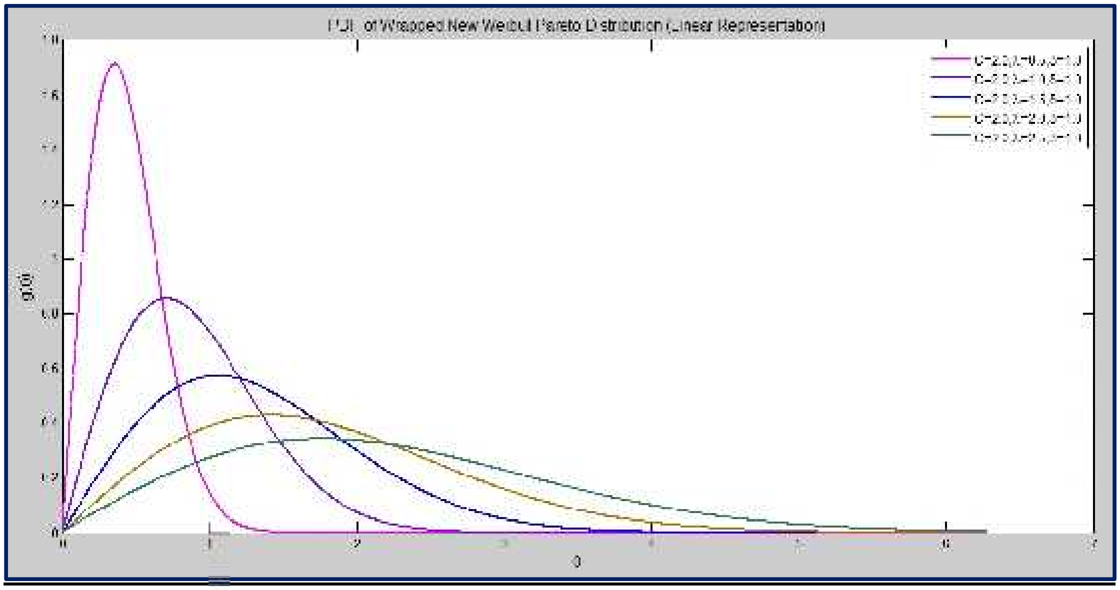


Figure 2: PDF of WNWP distribution (Linear Representation)

The same linear representation of pdf for different values of **λ** keeping the values for parameters ***c*** at 2.0 and ***δ*** at 1.0 is obtained as below



Now the graph depicting the circular representation of the pdf of WNWP Distribution for different values of **c, *δ***, keeping the value for the parameter **λ** at 2.0 is shown below:

Figure 3: PDF of WNWP distribution (Circular Representation)

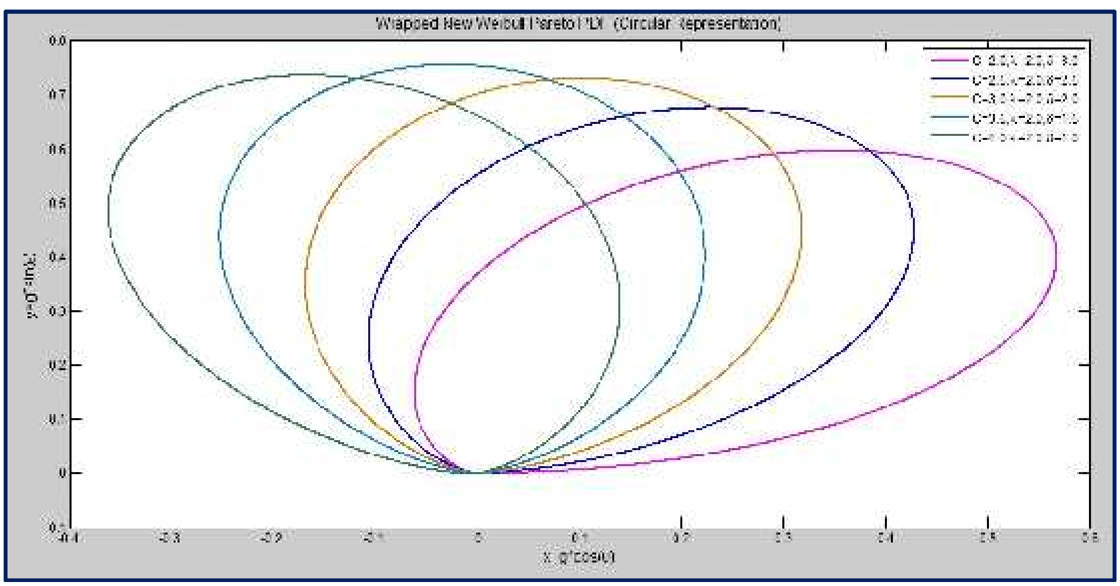
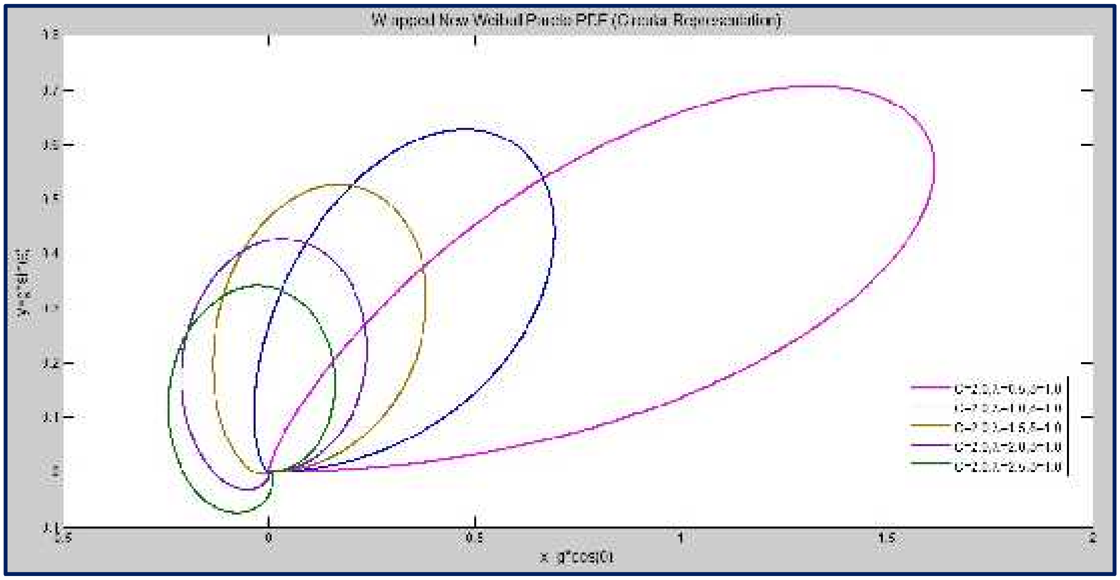


Figure 4: PDF of WNWP distribution (Circular Representation)

Same circular representation for the pdf now for the different values of λ keeping the values for parameters ***c*** at 2.0 and ***δ*** at 1.0 is obtained as below:



b) Cumulative Distribution Function for WNWP distribution:

The Distribution function of the WNWP distribution can be derived as

$$G(\theta) = \int_0^{\theta} \sum_{k=0}^{\infty} \frac{c.\delta}{\lambda}\left(\frac{\theta+2k\pi}{\lambda}\right)^{c-1} e^{-\delta\left(\frac{\theta+2k\pi}{\lambda}\right)^{c}} d\theta$$

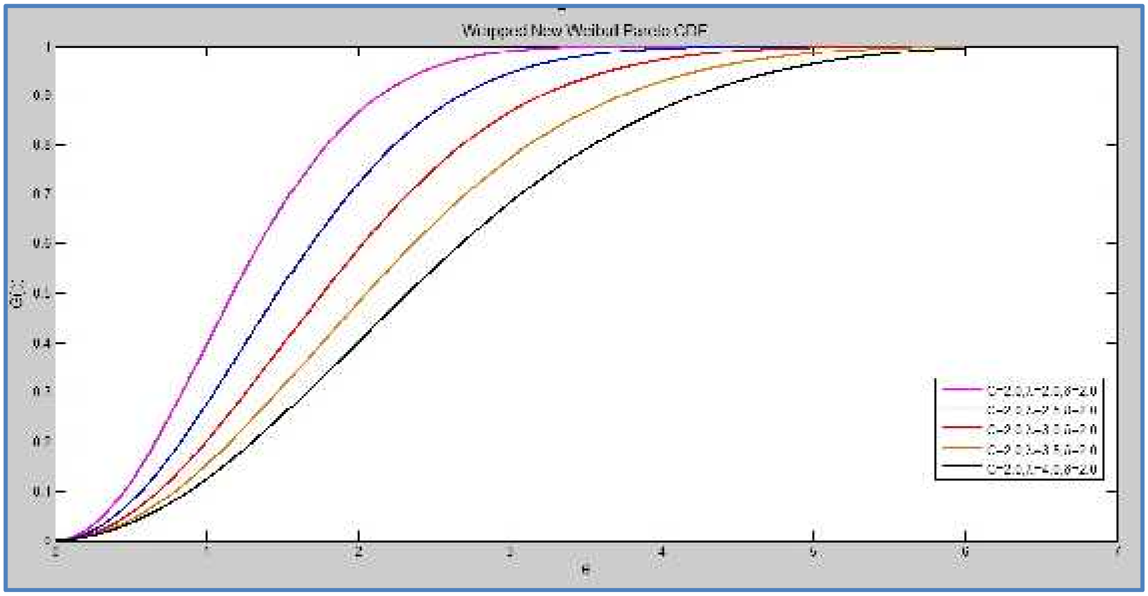
Taking $m=\left(\frac{\theta+2k\pi}{\lambda}\right)^{c}$ and solving the integral we get

$$G(\theta) = \sum_{k=0}^{\infty}\left[-e^{-\delta\left(\frac{\theta+2k\pi}{\lambda}\right)^{c}} + e^{-\delta\left(\frac{2k\pi}{\lambda}\right)^{c}}\right]$$

Where $\theta \in [0, 2\pi)$ and $c>0$, $\lambda>0$ and $\delta>0$

The graph for the CDF, $G(\theta)$ for WNWP distribution is obtained as below

Figure 5: CDF of WNWP distribution



c) Characteristic function for WNWP distribution:

As discussed in the previous section the characteristic function of NWP distribution is

$$¤(t) = \int_0^\infty e^{itx} \frac{c.¢}{a}\left(\frac{x}{a}\right)^{c-1} e^{-¢\left(\frac{x}{a}\right)^{c}} dx \qquad \ldots(4)$$

Taking $¢\left(\frac{x}{a}\right)^{c} = U$ we get

$$¤(t) = \int_0^\infty e^{ita(u/¢)^{(1/c)}} e^{-u} du$$

This can be written as

$$¤(t) = \int_0^\infty e^{-u} \sum_{k=0}^{\infty} \frac{\left(ita\left(\frac{u}{¢}\right)^{(1/c)}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty}\left(\frac{(ita)^k}{¢^{(k/c)}k!}\right)\int_0^\infty e^{-u} u^{\frac{k}{c}} du$$

$$= \sum_{k=0}^{\infty}\left(\frac{(ita)^k}{¢^{(k/c)}k!}\right)\Gamma(1+k/c) \qquad \ldots(5)$$

The convergence of the series in (5) fails at least for some values of c where $c = \frac{1}{n}, \quad n > 0, \quad n \in \mathbb{Z}^{+}$. To solve this for obtaining the trigonometric moments, the n – point Gauss – Laguerre quadrature formula for numerical integration as given in Rao et al, (1975) is applied. For $p \in \mathbb{Z}$ the characteristic function of the WNWP is

$$\varphi(p) = \int_0^{2\pi} e^{ipx} g(\theta)\, d\theta$$

The real and imaginary parts α_p and β_p respectively are obtained from the characteristic function of the WNWP distribution.

The following are the graphs for the characteristic function of the WNWP distribution showing the real part and imaginary part separately for different values of *c*, λ, and δ

Figure 6: Characteristic Function of WNWP distribution (1)

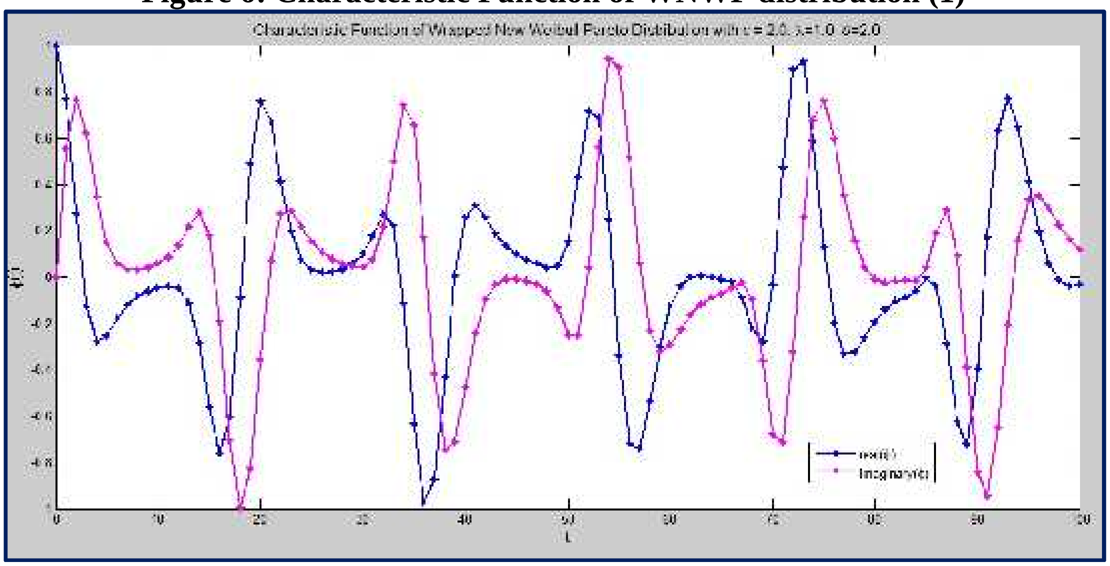
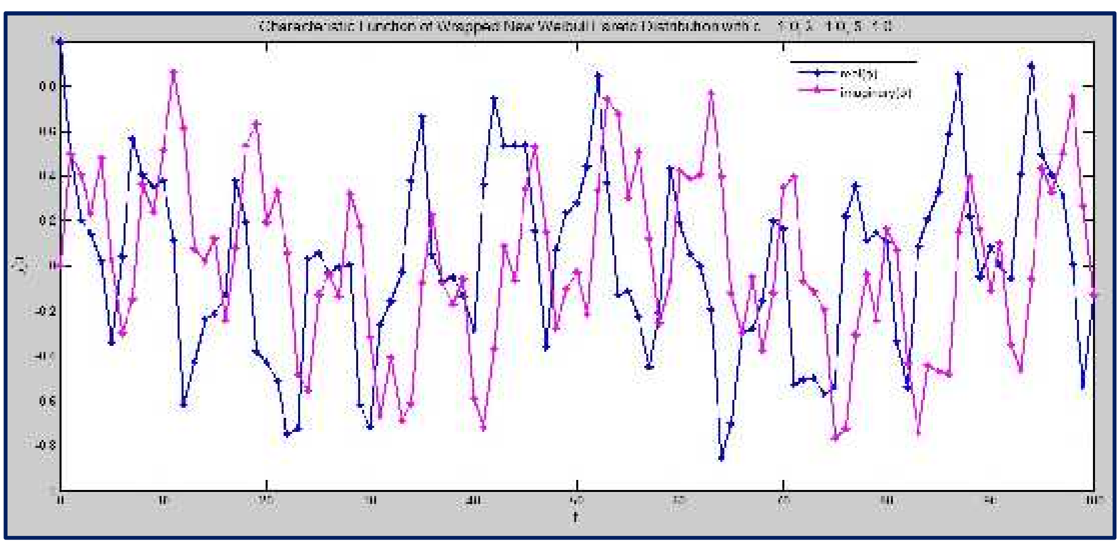


Figure 7: Characteristic Function of WNWP distribution (2)



d) Population Characteristics:

Given a Circular distribution, Mardia (2000) had derived expressions for mean direction $«_o$ resultant length $°_1$, Circular variance V_0, Central Trigonometric Moments $\sim^*_p$, $^*_p$, Skewness $¥^o_1$ and Kurtosis $¥^o_2$

Using these expressions the Population Characteristics for the New Weibull Pareto distribution for different values of the parameters *c*, λ and δ are computed and tabulated here under.

Table 1: Characteristics of New Weibull Pareto Distribution at λ = 3.0 and δ=2.0

c	c=0.5	c=1.5	c=2.0	c=2.5	c=3.0
Trigonometric Moments					
α_1	0.4600	0.0626	-0.1282	-0.2738	-0.3866
α_2	0.6280	-0.0982	-0.1795	-0.1978	-0.1781
β_1	0.3332	0.5630	0.6177	0.6410	0.6443
β_2	0.0271	0.1982	0.0601	-0.0822	-0.2163
Resultant Length					
ρ_1	0.5680	0.5665	0.6309	0.6971	0.7514
ρ_2	0.6286	0.2212	0.1893	0.2142	0.2802
Mean Direction					
μ_0	0.6269	1.4600	1.7754	1.9745	2.1113
Circular Variance					
V_0	0.4320	0.4335	0.3691	0.3029	0.2486
Circular Standard Deviation					
σ_0	1.0637	1.0661	0.9598	0.8496	0.7561
	0.9636	1.7372	1.8245	1.7555	1.5951
Central Trigonometric Moments					
α_1^*	0.5680	0.5665	0.6309	0.6971	0.7514
α_2^*	0.2214	0.1393	0.1408	0.1961	0.2747
β_1^*	0.0000	0.0000	0.0000	0.0000	0.0000
β_2^*	-0.5883	-0.1718	-0.1266	-0.0861	-0.0553
Skewness					
Υ_1^0	-2.0717	-0.6017	-0.5644	-0.5162	-0.4463
Kurtosis					
Υ_2^0	0.6289	0.1933	-0.1298	-0.4356	-0.7118

Table 1: Characteristics of New Weibull Pareto Distribution at c = 3.0 and λ= 2.0

δ	δ=0.5	δ=1.5	δ=2.0	δ=2.5	δ=3.0
Trigonometric Moments					
α_1	-0.4446	0.0107	0.1347	0.2255	0.2953
α_2	-0.1111	-0.5265	-0.5573	-0.5475	-0.5195
β_1	0.5730	0.8589	0.8720	0.8691	0.8598
β_2	-0.2041	0.0491	0.2075	0.3335	0.4324
Resultant Length					
ρ_1	0.7253	0.8589	0.8823	0.8979	0.9091
ρ_2	0.2324	0.5288	0.5946	0.6411	0.6759
Mean Direction					
μ_0	2.2308	1.5584	1.4176	1.3169	1.2400
Circular Variance					
V_0	0.2747	0.1411	0.1177	0.1021	0.0909
Circular Standard Deviation					
σ_0	0.8015	0.5515	0.5004	0.4642	0.4366
	1.7084	1.1289	1.0196	0.9430	0.8851
Central Trigonometric Moments					
α_1^*	0.7253	0.8589	0.8823	0.8979	0.9091
α_2^*	0.2253	0.5275	0.5939	0.6406	0.6755
β_1^*	0.0000	0.0000	0.0000	0.0000	0.0000
β_2^*	-0.0569	-0.0360	-0.0297	-0.0252	-0.0220
Skewness					
Υ_1^0	-0.3953	-0.6796	-0.7352	-0.7732	-0.8012
Kurtosis					
Υ_2^0	-0.6807	-0.8436	-0.8729	-0.8925	-0.9068

Table 1: Characteristics of New Weibull Pareto Distribution at c = 2.0 and δ = 3.0

λ	λ=0.5	λ=1.0	λ=1.5	λ=2.0	λ=2.5
Trigonometric Moments					
α_1	0.9589	0.8423	0.6685	0.4635	0.2547
α_2	0.8423	0.4635	0.0662	-0.1930	-0.2827
β_1	0.2515	0.4727	0.6391	0.7371	0.7647
β_2	0.4727	0.7371	0.7310	0.5477	0.3292
Resultant Length					
ρ_1	0.9913	0.9658	0.9249	0.8707	0.8060
ρ_2	0.9658	0.8707	0.7340	0.5807	0.4339
Mean Direction					
μ_0	0.2565	0.5114	0.7629	1.0095	1.2493
Circular Variance					
V_0	0.0087	0.0342	0.0751	0.1293	0.1940
Circular Standard Deviation					
σ_0	0.1319	0.2636	0.3951	0.5262	0.6567
	0.2636	0.5262	0.7865	1.0427	1.2922
Central Trigonometric Moments					
α_1^*	0.9913	0.9658	0.9249	0.8707	0.8060
α_2^*	0.9658	0.8706	0.7332	0.5772	0.4236
β_1^*	0.0000	0.0000	0.0000	0.0000	0.0000
β_2^*	-0.0016	-0.0115	-0.0332	-0.0634	-0.0940
Skewness					
Υ_1^0	-1.9695	-1.8274	-1.6152	-1.3628	-1.1001
Kurtosis					
Υ_2^0	0.4184	0.3522	0.2559	0.1468	0.0416

4 CONCLUSION

It can be observed that the Wrapped New Weibull Pareto distribution becomes Wrapped Exponential distribution when $c = \delta = 1$ and when $\delta = 1$, Wrapped New Weibull Pareto distribution reduces to wrapped Weibull distribution. Therefore while computing Population characteristics for the Wrapped New Weibull Pareto distribution these special cases are not considered.

From the population characteristics for the Wrapped New Weibull Pareto distribution tabulated above in the last section, we can observe that with increasing value of shape parameter ***c*,** keeping both scale parameters **δ = 2.0** and **λ = 3.0,** the Circular variance gradually decreased, the distribution incrementally reduced its negative skewness and remained platykurtic. With increasing value of one scale parameter $\boldsymbol{\delta}$ keeping other scale parameter λ at **2.0** and shape parameter ***c*** at **3.0**, the Circular variance gradually decreased, the distribution remained negatively skewed and platykurtic. Also with increasing value of scale parameter λ keeping other scale parameter $\boldsymbol{\delta} = \mathbf{3.0}$ and shape parameter ***c*** = **2.0**, the Circular variance gradually increased, the distribution reduced its negative skewness and remained platykurtic.

I. REFERENCES

1 Dattatreya Rao, A.V., Ramabhadra Sarma, I., Girija S.V.S., (2007). On wrapped version of some life testing models. Communication in Statistics - Theory and Methods, 36, 2027-2035.

2 Girija, S.V.S., 2010. Construction of New Circular Models. VDM - VERLAG, Germany.

3 Jammalamadaka S. Rao, Sen Gupta, A., 2001. Topics in Circular Statistics, World Scientific Press, Singapore.

4 Mardia, K.V. and Jupp, P.E. (2000), Directional Statistics, John Wiley, Chichester.

5 Ramabhadra Sarma, I., Dattatreya Rao, A.V. and Girija S.V.S., (2009). On Characteristic Functions of Wrapped Half Logistic and Binormal Distributions, International Journal of Statistics and Systems, Vol 4(1), pp. 33–45.

6 Ramabhadra Sarma, I., Dattatreya Rao, A.V. and Girija, S.V.S., (2011). On Characteristic Functions of Wrapped Lognormal and Weibull Distributions, Journal of Statistical Computation and Simulation Vol. 81(5), 579–589.

7 Rao, C.R. & Mitra, S.K. (1975), Formulae and Tables for Statistical Work, Statistical Publishing Society.

8 Tahir M. H, Gauss M. Cordeiro, Ayman Alzaatreh, M. Mansoor, and M. Zubair, A New Weibull-Pareto Distribution: Properties and Applications, Communications in Statistics - Simulation and Computation Vol. 45, Iss. 10,2016